# Lecture 5

Andrei Antonenko

February 10, 2003

## 1 Matrices

Now we'll start studying new algebraic object — matrices.

**Definition 1.1.** *The* **matrix** *is a (rectangular) table of the elements of* $\mathbb{R}$. *(Actually, we can consider matrices over fields other than* $\mathbb{R}$ *— in the future we will work with matrices over the field of complex numbers* $\mathbb{C}$.*)*

Now we'll introduce some notation that we will use. We will denote matrices with capital letters, and the elements of the matrix with same small letter with 2 subscripts, the first of them denotes the row, and the second one denotes the column. Often we will speak about $m \times n$-matrices, which means that it has $m$ rows and $n$ columns.

$$A = (a_{ij}), \quad A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

The matrix is called **square** matrix if the number of its rows is equal to the number of its columns. For every square matrix we will define its **main diagonal**, or simply **diagonal**, as a diagonal from the top left corner to the bottom right corner, i.e. diagonal consists of the elements $a_{11}, a_{22}, \ldots, a_{nn}$. Another diagonal is called **secondary**. It is used very rarely.

So, we introduced an object. But now we should introduce operations, otherwise the object is not interesting!

## 2 Matrix Operations

### 2.1 Addition

The first and the easiest matrix operation is matrix addition.

**Definition 2.1.** *Let $A$ and $B$ are $m \times n$-matrices. Then their* **sum** $C = A + B$ *is an $m \times n$-matrix such that $c_{ij} = a_{ij} + b_{ij}$, i.e. the elements of this matrix are sums of corresponding elements of the matrices $A$ and $B$.*

**Example 2.2.**

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 0 & -1 \end{pmatrix} + \begin{pmatrix} -2 & 0 & 3 \\ 1 & -1 & 3 \end{pmatrix} = \begin{pmatrix} -1 & 2 & 6 \\ 4 & -1 & 2 \end{pmatrix}$$

Now let's consider the properties of addition.

**(A1) Commutativity.** It is obvious that for any matrices $A$ and $B$ of the same size $A + B = B + A$.

**(A2) Associativity.** It is obvious that for any matrices $A$, $B$ and $C$ of the same size $(A + B) + C = A + (B + C)$.

Here we can mention that we can choose any order of matrices to perform the addition of 3 or more matrices. For example, we can prove, that $(A + B) + C = (A + C) + B$.

*Proof.*

$$\begin{aligned} (A + C) + B &= (C + A) + B && \text{commutativity} \\ &= C + (A + B) && \text{associativity} \\ &= (A + B) + C && \text{commutativity} \end{aligned}$$

$\square$

**(A3) Existence of the zero.** There exists a **zero matrix**

$$\mathbf{0} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

For any matrix $A$ we have that $A + \mathbf{0} = A$.

**(A4) Existence of the additive inverse.** For any matrix $A$ there exists matrix $-A$ such that $A + (-A) = \mathbf{0}$. The elements of this matrix $(-a)_{ij} = -a_{ij}$.

**Example 2.3.** *The additive inverse for the matrix* $\begin{pmatrix} 1 & 2 & -3 \\ 1 & 0 & -1 \end{pmatrix}$ *is* $\begin{pmatrix} -1 & -2 & 3 \\ -1 & 0 & 1 \end{pmatrix}$.

## 2.2 Multiplication by a number

For any matrix $A$ and for any number $c \in \mathbb{R}$ we can define the matrix $B = cA$, such that $b_{ij} = ca_{ij}$, i.e. we multiply all elements of the matrix $A$ by the same number $c$. This operation has the following obvious properties:

- $(c_1c_2)A = c_1(c_2A)$;
- $(c_1 + c_2)A = c_1A + c_2A$;
- $c(A + B) = cA + cB$.

## 2.3 Multiplication

The definition of multiplication is much more complicated than the definition of the previous operations.

**Definition 2.4.** *Let $A$ be an $m \times p$-matrix and $B$ be a $p \times n$-matrix. Then their* **product** *is an $m \times n$ matrix $C$ such that*

$$c_{ij} = \sum_{k=1}^{p} a_{ik}b_{kj}$$

So, we see, that in order to be able to multiply matrices, the number of columns of the first matrix should be equal to the number of rows of the second one.

**Example 2.5.**

$$\begin{aligned}
&\begin{pmatrix} 2 & 1 & -2 \\ 3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & 1 & 0 \end{pmatrix} \\
&\quad = \begin{pmatrix} 2\cdot 3 + 1\cdot 2 + (-2)\cdot 1 & 2\cdot 0 + 1\cdot 1 + (-2)\cdot 1 & 2\cdot 1 + 1\cdot 3 + (-2)\cdot 0 \\ 3\cdot 3 + 0\cdot 2 + 1\cdot 1 & 3\cdot 0 + 0\cdot 1 + 1\cdot 1 & 3\cdot 1 + 0\cdot 3 + 1\cdot 0 \end{pmatrix} \\
&\quad = \begin{pmatrix} 6 & -1 & 5 \\ 10 & 1 & 3 \end{pmatrix}
\end{aligned}$$

*We can see, that we cannot multiply these 2 matrices in different order, i.e. we can not compute*

$$\begin{pmatrix} 3 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 1 & -2 \\ 3 & 0 & 1 \end{pmatrix}$$

**Example 2.6 (Cute example of matrix multiplication).** *Let $\alpha$ and $\beta$ be real numbers, and let's compute the following product*

$$\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix}$$

*It is equal to:*

$$\begin{pmatrix} \cos\alpha\cos\beta - \sin\alpha\sin\beta & -\cos\alpha\sin\beta - \sin\alpha\cos\beta \\ \sin\alpha\cos\beta + \cos\alpha\sin\beta & -\sin\alpha\sin\beta + \cos\alpha\cos\beta \end{pmatrix} = \begin{pmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) \end{pmatrix}$$

*So,we can see that we get a matrix of the same type, but instead of $\alpha$ and $\beta$ we have $\alpha + \beta$.*

Now let's consider the properties of multiplication.

**(M1) Commutativity.** Unfortunately, commutativity does not hold for matrix multiplication. Moreover, for some matrices $A$ and $B$ we can compute $AB$ and cannot compute $BA$. E.g., if $A$ is a $2 \times 3$-matrix, and $B$ is a $3 \times 3$-matrix, then $AB$ is defined, and $BA$ is not. Now, we can give a counterexample even if both products are defined.

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \text{ but } \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Moreover, from this example we see that the product of two nonzero matrices can be a zero matrix.

**(M2) Associativity.** Associativity holds for matrix multiplication, i.e. for any three matrices such that all needed products(i.e., $AB$ and $BC$) can be defined, we have that $(AB)C = A(BC)$.

*Proof.* Let $A = (a_{ij}), B = (b_{ij}), C = (c_{ij})$. Then $(AB)_{ik} = \sum_l a_{il}b_{lk}$, and

$$((AB)C)_{ij} = \sum_k (AB)_{ik}c_{kj} = \sum_k \sum_l a_{il}b_{lk}c_{kj}$$

In the same way $(BC)_{lj} = \sum_k b_{lk}c_{lj}$, and so

$$(A(BC))_{ij} = \sum_l a_{il}(BC)_{lj} = \sum_l a_{il} \sum_k b_{lk}c_{kj}$$

Now we can change the order of the summation, and see that these expressions are equal. ∎

Here, unlike in the case of addition, we cannot choose any order, since commutativity does not hold for multiplication. For example,

$$(AB)C \neq (CA)B, \quad \text{etc.}$$

The order of the matrices one multiplies should be always preserved.

**(M3) Existence of the one.** There exists an **identity matrix**

$$I = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

This $n \times n$-matrix has 1's on its main diagonal. For any $m \times n$-matrix $A$ we have that $AI = IA = A$.

*Proof.* Can be done directly from the definition of the matrix multiplication. Simply can check that

$$AI = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

By the same arguments, $IA = A$

So, this matrix plays the same role for matrices as a number 1 for numbers. By multiplying by identity matrix, we do not change the given matrix. □

Next lecture we are going to talk about the multiplicative inverse for matrices.